Immense Integers - Solution

This problem is a more general version of the relatively well-known problem of finding positive integer solutions to

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 4,$$

which is a special case of this problem, namely for $t = 0$. We first illustrate how to solve this problem. Note that the entire equation is homogeneous, i.e. if we multiply all variables by λ, we still have a solution to the problem. We exploit this by multiplying the solution by c^{-1}. Then we want $a' = ac^{-1}$ and $b' = bc^{-1}$ so that

$$\frac{a'}{b'+1} + \frac{b'}{1+a'} + \frac{1}{a'+b'} = 4.$$

Of course, now a' and b' can be arbitrary rationals instead of integers, but we have removed one of the variables. To go from a solution of this equation to an integer solution, pick c as the least common multiple of the denominators of a' and b' and multiply $a = a'c$ and $b = b'c$. What is nice about our new equation is that it describes a plane curve in $\mathbb{R}^2$, on which we look for a positive rational point. In fact, this equation describes an *elliptic curve*, and the problem of looking for rational points on elliptic curves is well-studied. We first translate the elliptic curve to a more standard form, namely $y^2 = x^3 + Ax^2 + Bx$. It takes some computation, but one can derive that $A = 4(t^2 + t + 4)^2 + 12(t^2 + t + 4) - 3$ and $B = 32(t^2 + t + 7)$. Now, to find a positive rational point on the elliptic curve, the idea is to start with any rational point on the curve, and then add this point to itself until a positive point is found. With addition we do not mean the normal coordinate-wise addition, but instead the addition as defined on an elliptic curve. To add two points on an elliptic curve, draw a line through the two points and find the third intersection point of the line with the elliptic curve. Flip this point in the x-axis. As the curve is symmetric around the x-axis, this point still lies on the curve. This is the point representing the addition of the two starting points. To add a point to itself, instead take the tangent line to that point. This procedure will always yield a rational point when starting with rational points. Thus, we can keep iterating until a positive rational point is found.

The only issue is that we need a rational point on the curve to start with. In other words, we need an initial solution to the equation, which can be negative, but still needs to be rational. In the well-known problem where $t = 0$, a brute force check is enough to yield the relatively small solution of $(a, b, c) = (-1, 4, 11)$. This can be used to find an initial rational point on the transformed curve. However, in this more general case, it is very difficult to find an initial solution. After much calculation, such a point can in fact be found, exploiting the very specific form of the right hand side. For the complete expression of this point, see the provided solution code.

Even with this initial point and iteration, finding the solution can take a long time, hence why $0 \leq t < 4$. Still, the iteration for $t = 2$ takes too long and will yield a timelimit. Luckily, this case was given as a sample; the idea is to hardcode this case to achieve the timelimit ;)